International Workshop

Variational Modeling in Solid Mechanics

Program & Abstracts

September 22 – 24, 2014

University of Udine

Udine, Italy

Monday 22

08:30-09:00 Registration
09:00-09:15 Opening

09:15-10:15 Gianni Dal Maso. *Quasistatic evolution as limit of dynamic evolutions: the case of perfect plasticity*
10:20-10:50 Jean-Francois Babadjian. *Fracture versus delamination of thin films*

10:50-11:20 Coffee break

11:20-11:50 Dorothee Knees. *Global spatial regularity results for elasticity models with damage or friction*
12:00-12:30 Elisa Davoli. *Stress regularity for a new quasistatic evolution model of perfectly plastic plates*

12:30-15:00 Lunch break

15:00-16:00 Robert Lipton. *Understanding dynamic brittle fracture as a macroscopic limit of unstable mesoscopic dynamics*

16:00-16:30 Coffee break

16:30-17:00 Blaise Bourdin. *Some fracture problems originating from reservoir stimulation*
17:10-17:40 Lorenzo Freddi. *Residually stressed beams by Gamma-convergence*

20:00 Social dinner

Tuesday 23

09:00-10:00 Alain Goriely. *The differential geometry of nonlinear mechanics*
10:05-10:35 Anna Pandolfi. *Visco-hyperelastic models of electro-active biological tissues*

10:35-11:00 Coffee break

11:00-11:30 Marita Thomas. *Rate-independent, partial damage in thermo-viscoelastic materials with inertia*
11:40-12:10 Alessandro Giacomini. *Optimization problems for elastically supported membranes*

12:10-15:00 Lunch break

15:00-16:00 Andrea Braides. *Variational problems for antiferromagnetic systems*
16:05-16:35 Riccarda Rossi. *Existence results for rate-dependent elastoplasticity at finite strain*

16:35-17:00 Coffee break

17:00-17:30 Alessandro Musesti. *On a hyperelastic model of the skin*
17:40-18:10 Patrizio Neff. *A canonical extension of Korn's first inequality to H(Curl) motivated by gradient plasticity with plastic spin*

Wednesday 24

09:00-10:00 Irene Fonseca. *Variational methods for crystal surface instability*
10:05-10:35 Giuseppe Tomassetti. *Dissipative size effects in strain-gradient plasticity: the case of simple shear*

10:35-11:00 Coffee break

11:00-12:00 Cesare Davini. *Composite thin-walled beams by Gamma-convergence*

Fracture versus delamination of thin films

Jean-François Babadjian

Université Paris 6 - Pierre et Marie Curie

This talk is devoted to highlight the interplay between fracture and delamination in thin films. The usual scaling law on the elasticity parameters and the toughness of the medium with respect to its thickness gives rise to traditional cracks which are invariant in the transverse direction. We will show that, upon playing on this scaling law, it is also possible to observe debonding effects (delamination as well as decohesion) through the appearance of cracks which are orthogonal to the thin direction. Starting from a three-dimensional brittle elastic thin film, we will first present how both phenomena can be recovered independently through a Gamma-convergence analysis as the thickness tends to zero. Then, working on a "toy model" for scalar anti-plane displacements, we will show how both phenomena can be obtained at the same time. Some partial results in the full three-dimensional case will be presented. This is a joint work with A. A. Leon Baldelli, B. Bourdin, D. Henao and C. Maurini.

Some fracture problems originating from reservoir stimulation

Blaise Bourdin

Louisiana State University

Recent progress in subsurface energy extraction, such as gas shales or enhanced geothermal system, rely on advanced reservoir stimulation techniques in order to create networks of highly interconnected cracks. Due to their complexity, rigorous modeling of these techniques is well beyond reach of classical fracture models.

In this talk, I will focus on hydraulic and thermal fracturing. I will present simplified yet realistic models derived from Francfort and Marigo's variational approach to fracture. In the setting of hydraulic fracturing ("fracking"), I will first focus on the simple problem of an isolated crack in an infinite isotropic homogeneous reservoir for which a close form solution is available. I will present a numerical approach based on regularized fracture energy, and show verification numerical simulations. I will then present numerical simulations suggesting that some some hypotheses commonly made in industry may be incorrect. I will then focus on a problem related to thermal stimulation ("cryofrac"). In the process of providing validation and verification of the variational approach applied to this problem, I will highlight how the now classical debate over local vs. global minimization may be irrelevant and propose an alternative approach.

Variational problems for antiferromagnetic systems

Andrea Braides

Università "Tor Vergata", Roma

Antiferromagnetic spin systems exhibit some features, as frustration or pattern formation, that bring some interesting issues in their continuum approximation. I will present some questions and (few) answers on this subject.

Quasistatic Evolution as Limit of Dynamic Evolutions: the Case of Perfect Plasticity

Gianni Dal Maso

S.I.S.S.A., Trieste

We introduce a model of dynamic visco-elasto-plastic evolution in the linearly elastic regime and we prove an existence and uniqueness result. Then we study the limit of (a rescaled version of) the solutions when the data vary slowly. We prove that they converge, up to a subsequence, to a quasistatic evolution in perfect plasticity. This is a joint work with R. Scala (SISSA).

Composite thin-walled beams by Γ-convergence

Cesare Davini

Università di Udine

The behavior of thin-walled beams does not fit the De Saint-Venants theory of beam and a multitude of ad hoc models have been proposed throughout the years, starting from that of Vlasov. We consider a beam whose cross-section is a tubular neighborhood of a simple curve γ. The two instances that the curve is either open or closed are considered. We assume that the wall thickness of the walls scales with a parameter δ_ε, while the length of γ scales with ε, and characterize a thin-walled beam for which δ_ϵ goes to zero faster than ε. Starting from the three dimensional linear theory of elasticity and letting ε go to zero, we derive a one-dimensional Γ-limit problem for the case in which the ratio between ε^2 and δ_ε is bounded. The limit model is obtained for a fully anisotropic and inhomogeneous material, thus making the theory applicable for composite thin-walled beams. The approach recovers in a systematic way, and gives account of, many features of the beam models in the theory of Vlasov. This is a joint work with L. Freddi and R. Paroni.

Stress regularity for a new quasistatic evolution model of perfectly plastic plates

Elisa Davoli

Carnegie Mellon University

In this talk we will dicuss some properties of solutions to a quasistatic evolution problem for perfectly plastic plates, that has been recently derived from three-dimensional Prandtl-Reuss plasticity. We will prove that the stress tensor has locally square-integrable first derivatives with respect to the space variables. We will also exhibit an example showing that the model under consideration has in general a genuinely three-dimensional nature and cannot be reduced to a two-dimensional setting. The talk is based on a joint work with Maria Giovanna Mora.

Variational Methods for Crystal Surface Instability

Irene Fonseca

Carnegie Mellon University

Using the calculus of variations it is shown that important qualitative features of the equilibrium shape of material voids and quantum dots in a linearly elastic solid may be deduced from smoothness and convexity properties of the interfacial energy.

In addition, short time existence, uniqueness, and regularity for an anisotropic surface diffusion evolution equation with curvature regularization are proved in the context of epitaxially strained two-dimensional films. This is achieved by using the H^{-1}-gradient flow structure of the evolution law, via De Giorgi's minimizing movements, and it seems to be the first short time existence result for a surface diffusion type geometric evolution equation in the presence of elasticity.

Residually stressed beams by Γ-convergence

Lorenzo Freddi

Università di Udine

The theory of linear elasticity with residual stress goes back to Cauchy (1829), but for a long time the attention of researchers was almost exclusively given to the so-called linear theory of elasticity. In recent years, instead, the theory with residual stress has been studied and used quite extensively. As in the case of the derivation by Γ-convergence of thin structures without residual stress, initiated 20 years ago, also the corresponding asymptotic analysis for a material with residual stress started in 2006 from the simplest case of a plate with the paper of Paroni [3]. After that, in 2009, Della Longa and Londero [1] have considered the case of a thin-walled beam with a rectangular cross-section. The most difficult case of a slender rod, which is the subject of this talk, has been recently studied in [2].

The presence of residual stress introduces in the constitutive equation for the Piola-Kirchhoff stress tensor a dependence from the displacement gradient and not simply on the strain as in the case without residual stress. Precisely, the Piola-Kirchhoff stress tensor S is given by

$$S = \mathring{T} + Du\mathring{T} + \mathbb{L}Eu,$$

where Du denotes the gradient of the displacement u, Eu is the symmetric part of Du, $\mathring{T}$ is a second order symmetric tensor representing the residual stress in the reference configuration and $\mathbb{L}$ is a fourth order tensor called incremental elasticity tensor. The term $Du\mathring{T}$, that comes into play because of material frame indifference, makes the theory quite different from the elastic theory without residual stress; for instance, the elastic energy density is no longer convex.

In our analysis we do not impose any material symmetry on the incremental elasticity tensor $\mathbb{L}$ and we allow it to depend on the longitudinal variable y_3, i.e., the cross-sections of the beams are assumed to be homogeneous. By assuming the rod to be clamped to one of its bases, we find that the elastic energy of the limit problem is

$$I_{1\mathrm{d}}(\xi,\theta) = \frac{1}{2}\int_0^\ell Q(y_3,\xi_1'',\xi_2'',\xi_3',\theta') + \mathrm{tr}\langle\mathring{T}\rangle\,\theta^2 + \langle\mathring{T}\rangle(\xi_1',\xi_2')\cdot(\xi_1',\xi_2')\,dy_3,$$

where ξ is a Bernoulli-Navier displacement and θ is a scalar field representing the twist of the cross-section around the longitudinal axis. The energy density Q is defined by a minimum problem on the cross-section involving the incremental elasticity tensor $\mathbb{L}$, and

$$\langle\mathring{T}\rangle := \int_\omega \begin{pmatrix} \mathring{T}_{11} & \mathring{T}_{12} \\ \mathring{T}_{21} & \mathring{T}_{22} \end{pmatrix} dy_1 dy_2,$$

where ω denotes the cross-section.

References

[1] L. Della Longa and A. Londero, *Thin walled beams with residual stress*, J. Elasticity **96** (2009), 27–41.

[2] L. Della Longa, L. Freddi, A. Londero and R. Paroni, *Residually stressed beams*, Mathematics and Mechanics of Solids. Doi: 10.1177/1081286512454448.

[3] R. Paroni, *Theory of linearly elastic residually stressed plates*, Math. Mech. Solids **11** (2006), 137–159.

Optimization problems for elastically supported membranes

Alesssandro Giacomini

Università di Brescia

Shape optimization problems for domains whose state is determined by an elliptic equation with Robin boundary conditions (the case of elastically supported membranes) are recast in the framework of free discontinuity problems. A suitable class which guarantees the existence of an optimal domain is suggested by regularity properties of SBV-minimizers of a free discontinuity functional with (unusual) trace dependent surface term. This is a joint work with Dorin Bucur.

The Differential Geometry of Nonlinear Mechanics

Alain Goriely

University of Oxford

There is a well-known connection between differential geometry and nonlinear elasticity dating back to the work of Kondo, Bilby, Kroner, Davini among others. The basic idea is to use a non-Euclidean description of the reference configuration to describe continuous distributions of anelastic effects appearing in anelastic theories such as defects mechanics, thermo-elasticity, of morpho-elasticity. In this talk, I will first review the basic descriptors of non-Euclidean geometry (metric, curvature, torsion, connection and non-metricity) and use them to classify the geometry of the material manifold according to the type of defect encountered. Then, I will show how to transform these concepts into practical tools by using methods of differential geometry (Cartan's moving frames) to solve explicitly problems of nonlinear anelasticity. In particular, I will show how to compute exactly residual stress fields for systems with discombinations, that is combinations of distributed dislocations, disclinations, and point defects. This is joint work with Arash Yavari from Georgia Tech.

Global Spatial Regularity for Elasticity Models with Damage or Friction

Dorothee Knees

University of Kassel, Germany

For the analysis of strongly coupled material models it is useful to have deeper insight into the spatial regularity properties of the involved quantities like displacement fields or internal variables. In this lecture we will discuss some recent results for non-smooth situations with a special focus on Tresca friction models and on certain rate-independent damage models in the small strain regime. The talk relies on joint results with R. Rossi (University of Brescia), C. Zanini (Politecnico di Torino) and A. Schroeder (University of Salzburg).

Understanding Dynamic Brittle Fracture as a Macroscopic Limit of Unstable Mesoscopic Dynamics

Robert Lipton

Louisiana State University

The dynamic fracture of brittle solids is a particularly interesting collective interaction connecting both large and small length scales. Apply enough stress or strain to a sample of brittle material and one eventually snaps bonds at the atomistic scale leading to fracture of the macroscopic specimen.

With these ideas in mind we investigate a new class of models for solving problems of free crack propagation described by the peridynamic formulation. In the peridynamic formulation material points interact through short-range forces acting over a prescribed horizon. The formulation allows for discontinuous deformations associated with cracks. We choose a peridynamic model for which the short-range forces between material points are unstable and soften beyond a critical relative displacement. This model is used to represent the dynamics at mesoscopic length scales. We upscale the mesoscopic dynamics to identify the macroscopic dynamics as viewed across coarser length scales. An analysis shows that the associated macroscopic evolution has bounded energy given by the bulk and surface energies of classic brittle fracture mechanics. The macroscopic free crack evolution corresponds to the simultaneous evolution of the fracture surface and linear elastic displacement away from the crack set. The elastic moduli, wave speed, and energy release rate for the macroscopic evolution are explicitly determined by moments of the peridynamic potential energy. This delivers a new connection between nonlocal short-range forces acting over small length scales and dynamic free crack evolution inside a brittle medium as observed at the macroscopic scale. The analysis supports the use of appropriately calibrated nonlocal models for dynamic brittle fracture computation.

On a hyperelastic model of the skin

Alessandro Musesti

Università Cattolica Brescia

A celebrated class of elastic energy densities modeling the skin of a rabbit, introduced by Tong and Fung in 1976, involves an exponential term. We will prove existence of the solution for the associated variational problem with a general measure-valued external load. Moreover we investigate some properties of the minimizers, focusing in particular on the validity of the associated Euler-Lagrange equation.

A Canonical Extension of Korn's First Inequality to $\mathsf{H}(\mathrm{Curl})$ motivated by Gradient Plasticity with Plastic Spin

Patrizio Neff

Universität Duisburg-Essen

We prove a Korn-type inequality in $\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times3})$ for tensor fields P mapping Ω to $\mathbb{R}^{3\times3}$. More precisely, let $\Omega\subset\mathbb{R}^3$ be a bounded domain with connected Lipschitz boundary $\partial\Omega$. Then, there exists a constant $c>0$ such that

$$c\|P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}\leq\|\operatorname{sym}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}+\|\operatorname{Curl}P\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times3})}\qquad(1)$$

holds for all tensor fields $P\in\overset{\circ}{\mathsf{H}}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times3})$, i.e., all $P\in\mathsf{H}(\mathrm{Curl};\Omega,\mathbb{R}^{3\times3})$ with vanishing tangential trace on $\partial\Omega$. Here, rotation and tangential trace are defined row-wise. For compatible P, i.e., $P=\nabla v$ and thus $\operatorname{Curl}P=0$,

where $v \in \mathsf{H}^1(\Omega, \mathbb{R}^3)$ are vector fields having components v_n, for which ∇v_n are normal at $\partial\Omega$, the presented estimate (1) reduces to a non-standard variant of Korn's first inequality, i.e.,

$$c\|\nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})} \leq \|\operatorname{sym} \nabla v\|_{\mathsf{L}^2(\Omega,\mathbb{R}^{3\times 3})}.$$

On the other hand, for skew-symmetric P, i.e., $\operatorname{sym} P = 0$, (1) reduces to a non-standard version of Poincaré's estimate. Therefore, since (1) admits the classical boundary conditions our result is a common generalization of the two classical estimates, namely Poincaré's resp. Korn's first inequality. Applications to infinitesimal gradient plasticity with plastic spin are given.

Visco-hyperelastic Models of Electro-active Biological Tissues

Anna Pandolfi

Politecnico di Milano

The behavior of fiber reinforced active tissues, namely the excitation-contraction coupling (typical of myocardium, intestine, vascular walls, etc.), is due basically to the nonlinear interplay between the passive elastic tissue (mainly the extracellular collagen) and the active muscular network. The observed macroscopic dynamics derives as the emergent behavior of a complex multiscale architecture spanning several length scales: from the nanometer molecular actin-myosin calcium mediated interaction, going through the micrometer single cell contraction, up to the centimeter scale of the whole organ. Experimental evidences have demonstrated that the loading rate strongly affects the macroscopic behavior of biological electro-active systems. For example, cardio-myocytes are able to adjust their contractile strength according to the beating rate. Despite the wide literature on the subject, a few studies explored the role of stresses induced by the viscosity on the electromechanical coupling. In this work we refer to a fiber reinforced EA material model of intestine. We consider an extended MEF behavior and introduce the thermal coupling via a generalized thermodynamic approach that accounts also for viscosity [1]. In particular we provide the general theoretical framework of the electromechanical problem with time-dependent behaviors. We derive the constitutive relationships by assuming a decoupled expression of the Helmholtz free energy density and the multiplicative decomposition of the

deformation gradient. We specialize the formulation to a simplified phenomenological material model, validating the response against experimental measurements and providing a few representative numerical applications. This is a joint work with A. Gizzi, C. Cherubini, and S. Filippi (Campus Biomedico, Roma).

REFERENCES

[1] Q. Yang, L. Stainier, and M. Ortiz. A variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids. J. Mech. Phys. Sol. Vol. 54 pp. 401-424 (2006).

Existence results for rate-dependent elastoplasticity at finite strain

Riccarda Rossi

Università di Brescia

We address a model for rate-dependent finite-strain elastoplasticity. Existence results are proved passing to the limit in a time-incremental minimization scheme via variational convergence techniques. This is a joint work with Alexander Mielke and Giuseppe Savaré.

Rate-independent, partial damage in thermo-viscoelastic materials with inertia

Marita Thomas

WIAS, Berlin

This contribution deals with a model for rate-independent, partial damage in visco-elastic materials with thermal and inertial effects. In the spirit of Generalized Standard Materials, the damage process is modeled by means of an internal variable, governed by a rate-independent flow rule. The latter is coupled in a highly nonlinear way with the heat equation and the momentum balance for the displacements. We present a suitable weak formulation and give an existence result obtained with the aid of a partly decoupled time-discrete scheme and variational convergence methods. For this model we also discuss the asymptotic analysis for vanishing viscosity and inertia, which

leads to a fully rate-independent limit model for displacements and damage that is independent of temperature.

This is joint work with Giuliano Lazzaroni (SISSA Trieste), Riccarda Rossi (University of Brescia), and Rodica Toader (University of Udine).

Dissipative size effects in strain-gradient plasticity: the case of simple shear

Giuseppe Tomassetti

Università “Tor Vergata”, Roma

Metallic components undergoing inhomogeneous plastic flow display size-dependent behavior in the size range below 100μm, with smaller components being harder and having higher relative strength. This behavior is captured by strain-gradient plasticity theories, whose free energy and dissipation incorporate material length scales through a dependence on the gradient of plastic strain. In previous work [M. Chiricotto, L. Giacomelli, GT, SIAM J. Appl. Math., 72 (2012), 1169-1191] we have considered the rate-independent case of a theory of strain-gradient plasticity devised in [M. Gurtin, L. Anand J. Mech. Phys. Solids 53 (2005) 1624-1649] in the setting of small strains and we have quantified the influence of the energetic scale on hardening by a careful analysis of the solutions of a quasistatic evolution problem that mimics torsion experiments. Our ongoing research is now focusing on the dissipative scale, which is known to affect size-dependent strengthening: the smaller the sample, the higher the critical load which triggers plastic flow. In order to quantify the effect of the energetic scale on strengthening, we consider a rate-independent evolution problem that describes simple shear of an infinite strip [L. Anand et al, J. Mech. Phys. Solids 53 (2005) 1789-1826]. For this problem we can rigorously prove that smaller samples are stronger and we can determine the dependence of the critical load on the energetic scale. Three points are of special importance in our analysis. The first point is that the concept of energetic solution [A. Mielke, F. Theil, NoDEA 11 (2004) 151-189] can be effectively used to characterize quasistatic evolution processes in plasticity, both conventional [G. Dal Maso, A. DeSimone, M.G. Mora, Arch. Ration. Mech. Analysis 180 (2006) 237-291] and of strain-gradient type [A. Giacomini, L. Lussardi, SIAM J. Math. Anal. 40 (2008) 1201-1245]. Using this characterization it is easy to see that the hallmark of the onset of plastic flow is the instability of the trivial configuration, which is equivalent to saying that the minimum of a certain ”stability functional” becomes negative. The

second point is the observation that dissipation is (positively) homogeneous of degree one whereas free energy is quadratic. Then, a simple scaling argument can be used to show that free energy does not affect the critical load. Accordingly, we may replace the stability functional with a "reduced functional" which contains only contributions from applied loads and dissipation. The third point is that the reduced functional does not admit a minimum in the function space H^1_0, the state space of energetic solutions. Indeed, since dissipation is homogeneous of degree one, the reduced functional has linear growth and hence minimizers must be sought in BV. This notwithstanding, a result from [M. Amar, M. Chiricotto, L. Giacomelli, G. Riey, J. Math. Anal. Appl. 397 (2013), 381-401] tells us that the minimizer (unique in SBV) is smooth and solves the Euler-Lagrange equation. After some manipulations of the E-L equation, we are eventually able to determine the dependence of the critical load on the dissipative length scale.